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XV.

ANTICIPATION OF THE LISSAJOUS CURVES.

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Presented January 12, 1881.

IN 1857, Lissajous communicated to the French Academy of Sciences his " Mémoire sur l'Étude Optique des Mouvements Vibratoires." * By attaching a mirror to each of two tuning-forks, and reflecting a pencil of light successively from the two mirrors, the resultant motion of the forks was exhibited on a greatly amplified scale. When the vibrations of the tuning-forks were in parallel planes, and the musical interval between them was not exactly in tune, the consequent beats were manifested to the eye, even after they had become insensible to the ear. When the two motions were in rectangular planes, the extremity of the pencil of light described some one of a group of curves, rising to higher orders as the ratio which expressed the musical interval became less simple. Each of these groups could be easily resolved into eight distinct varieties, each variety depending on a definite difference of phase in the two movements. Inasmuch, however, as there are innumerable differences of phase between any two well-marked limits, there must be an endless series of curves in each group, through which each variety flows gradually into the next. Lissajous studied the subject analytically as well as experimentally, and then applied it to his optical method of tuning notes, especially those of tuning forks, to any required interval. If the vibrations of the forks were true to the precise ratio of this interval, the resultant orbit of the twice-reflected ray would be invariable: and it would be one or another variety in the corresponding group of curves, as the original difference of phase was smaller or larger. If the movements of the two forks did not conform to the exact ratio, the original difference of phase would change with each successive vibration, and the whole group of curves would be presented in succession. From the rapidity of this succession, the rate at which the phase changed would be known, and hence the degree of imperfection in the assumed interval.

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* Annales de Chim. et de Phys., vol. li. 1857.

The admiration of physicists was elicited by Lissajous's method of magnifying the mere trembling of a rigid tuning-fork into a visible magnitude; and especially by his optical method of tuning, which, in the hands of even a deaf person, would give better results than were possible to the nicest musical ear. Of these groups of beautiful curves, a few, as the circle, ellipse, parabola, and lemniscate, were not unfamiliar to mathematical and physical science. But most of them were supposed to be novel, and have been introduced into later works on Mechanics and Acoustics under the name of the "Lissajous curves." Many modifications have been introduced into the original experiments of Lissajous, either by himself or others. Reeds, driven by a bellows, have been substituted for tuning-forks. A steel rod, with an appropriate cross-section and mounted with a mirror, admits of the two elementary vibrations in rectangular planes. Two vibrating disks, with rectangular slits, which allow the light to pass only at their intersection, answer the same purpose. Sometimes one vibration is given to a small aperture through which the light passes, and the other to a lens which forms its image on a screen or in the eye. Wheatstone produced, in a single reflector, the two rectangular movements by mechanical means. König, and more recently Ritchie, have used mechanism with two mirrors. The tonophant of Ladd consists of a compound rod, flattened in two rectangular planes. Barrett succeeded with a round and bent steel wire.

It appears that in 1844, Professor Blackburn * of Glasgow experimented with a pendulum, in which the bob was suspended by a Y-shaped cord; the length of the pendulum being *virtually* the total length, or only that of the lower branch, in the two principal planes of vibration. In 1871, Mr. Hubert Airy † noticed the curious curves described by the end of a twig of acacia and of hazel. In pursuing the subject, he finally adopted Blackburn's compound pendulum, and reproduced many of the Lissajous curves; but he makes no allusion either to Lissajous or to Blackburn. Airy calls the orbits he obtained autographic curves, as they were neatly drawn by a pencil or pen, attached to the pendulum. Before and since that time, various machines were contrived for obtaining a permanent record of these complex and diversified motions, whether produced by the vibration of elastic bodies or by machinery. Lissajous and Desains, with an apparatus constructed by Froment, scratched the curves upon smoked glass, the style being moved by one vibrating body and the glass by the other. Professor Pickering[*] moved a pen and paper in rectangular directions by machinery, and described the curves on a much larger scale. Donkin[†] contrived a machine for the graphical representation of a number of parallel vibrations; but the most complete apparatus of the kind is Tisley's harmonograph,[‡] which can be suited to parallel, perpendicular, or oblique vibrations; and in a great variety of ratios. The motions of the pen and paper are produced by two independent pendulums, delicately mounted and heavily loaded, so as to maintain their vibration for a long time. The ratios can be altered by a change of weight, combined with a change in the length of one of the pendulums. The latter adjustment can be neatly applied, for making or disturbing the exact ratio of the required interval, without stopping the motion of the pendulum.

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* Tait's Dynamics of a Particle, 3d ed. p. 224.

† Nature, vol. iv. pp. 310 and 370.

As far back as 1800, Dr. Young[§] experimented upon the variegated path described by a single point of a silvered vibrating cord, illuminated by strong sunlight. In 1827,[‖] Wheatstone invented the kaleidophone: which was simply a vibrating wire, with a bead at the free end, and short enough to give persistent vision for the orbit. But neither Young nor Wheatstone have given a mathematical analysis of the motions; the exquisite figures they obtained were due, mostly, to the superposition of the higher harmonics, and few of them are identical with the Lissajous curves. In 1832, Edward Sang[¶] of Edinburgh developed mathematically the resultant of two rectangular vibrations, having different periods and phases, and illustrated his subject by experiments with round and flat wires, which produced the peculiar Lissajous curves. Drach, who himself, in 1846,[**] published his theoretical studies on the combination of two circular motions, with their resulting epicyclical curves, states that Perigal devised machinery which traced curves identical with those of Airy, and, therefore, with those of Lissajous. This machine was exhibited, in 1846, at a scientific *soirée* in the house of Lord Northampton. Three volumes of Perigal's kinematic curves are preserved in the archives of the Royal Society of London, of the Royal Astronomical Society, and of the Royal Society.

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[*] Journal of Franklin Institute, January, 1869.

[†] Proc. Roy. Soc. London, vol. xxii. (1874), p. 197.

[‡] Engineering, vol. xvii. (1874) p. 101.

[§] Trans. Roy. Soc., London (1800).

[‖] Collected Papers.

[¶] Edinb. Phil. Jour., vol. xii. (1832) p. 317.

[**] Phil. Mag., London, vols. xxxiv. pp. 448, 449 and xxxv. (1849–50).

But all these anticipations of the Lissajous curves, theoretical and experimental, are antedated by two publications which appeared at Boston in 1815.* In this year Professor Dean, of Burlington, Vermont, published a curious memoir on the "Motions of the Earth as seen from the Moon." It is well known that the time of the moon's rotation on its axis is equal to the time of its revolution in its orbit. The result of this equality (whether regarded as a mere coincidence or the effect of gravitation) is seen in the fact that the same side of the moon is always turned towards the earth. This statement would be literally true if the moon's motions of rotation and revolution were in parallel planes, and the *momentary* velocities as well as the *average* velocities were always equal. But the plane of the moon's equator is inclined to the plane of the moon's orbit at an angle of 6° 39′; so that an observer on the earth, at one time, overlooks the north pole of the moon to the extent of 6° 39′, and about a fortnight later the south pole. Moreover, the velocity of rotation is uniform, but the velocity of revolution is unequal in different parts of the elliptical orbit. Therefore, although the whole periods are the same for both motions, the moon at perigee revolves faster than it rotates, and at apogee it rotates faster than it revolves. Hence, in the course of one revolution, an observer on the earth overlooks the eastern or the western edge of the moon to the extent of 6° 18′ each way, or the greatest equation of the centre of the moon's orbit. These effects are familiar to astronomers under the name of the moon's librations, which enable them to see every month four sevenths of the moon's surface, though the remaining three sevenths are forever hidden from view.

Professor Dean investigates the influence of these peculiarities in the moon's motion on the position of the earth as seen by a lunar observer. If the two motions of the moon were equal and parallel, without any qualification, the earth would always occupy the same place in the lunar firmament. If the observer were happily situated at the centre of the moon's visible disk, the earth would always be in his zenith. But the inclination of the planes of the moon's equator and orbit will make the earth appear to oscillate over a north and south line, 13° 18′ in length. The want of uniformity in the velocity of the moon's revolution will produce an easterly and westerly oscillation in the earth's position, to the extent of 12° 36′. Nature here offers an astronomical example, on a grand scale, of two rectangular vibrations of nearly equal amplitude. It is the case of a unison, but a unison not exactly in tune. For the monthly period which governs both motions is not measured in exactly the same way for both. The period of the first oscillation is the time in which the moon revolves from a node to the same node again. The period of the second oscillation is the time occupied by the moon in going from perigee to perigee again. If the node and the perigee were fixed points, the two periods would be equal. But the node and perigee, in the mean while, are both moving; the former about 1½° to the west, and the latter about 3° to the east. Hence the periods of the two oscillations will be in the ratio of about 358½ to 363, or of 79½ to 80½. In the course of eighty oscillations, the north and south one would gain one whole oscillation upon the other; and once, in about every six years, the earth, as seen from the moon, would appear to describe upon the sky all the varieties in Lissajous's first group of orbits. To illustrate the subject, Dean devised the compound pendulum, supposed to have been introduced into science twenty-nine years afterwards by Blackburn. The total length of the cord was forty inches and that of the single branch thirty-nine inches. With these proportions, he realized the case of a unison at fault to the required degree; and he had the satisfaction of seeing the suspended bob move over, in the period of eighty vibrations, the complete series of paths pertaining to all the changes of phase of an imperfect unison.

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* Memoirs of Amer. Acad. of Arts and Sciences, 1st series, vol. iii. p. 241 (1815).

In the same year, an elaborate memoir was published by Dr. Nathaniel Bowditch,* "On the Motion of a Pendulum suspended from two Points," which begins with this paragraph, — "The remarkable variety of motions in a pendulum suspended from two points, in the curious experiment in Professor Dean's paper on the apparent motion of the earth, as viewed from the moon, induced me to examine the theory of such motions, and I have found the fluents of the fundamental equations, where the arcs of oscillation are small, which is the case usually considered in simple pendulums. Some of the most important results of this calculation are contained in the following articles."

Dr. Bowditch found that the equation which expressed the path described by the end of the pendulum was, in the general case, transcendental, but became algebraical when the periods of the component motions were in some simple ratio, either exactly or approximately. He developed in detail: 1. The unison; 2. The octave; 3. The twelfth; 4. The double octave: and all of them whether perfect or out of tune. His investigation had no reference to acoustics; and he describes these cases by their ratios, and not by the musical intervals which these ratios may represent. He adds that similar results would be obtained if the longer period, divided by the shorter, were expressed by any whole number, either exactly or approximately. Such ratios as 2 : 3 or 3 : 4, etc., did not come within the scope of his paper. Dr. Bowditch obtains the equations for the few cases which he specially examines, and finds that for the unison the path is expressed by an equation of the first or the second order; for the octave, by an equation of the second or fourth order; for the ratio 1 : 3, by an equation of the third or the sixth order; and for the double octave, by an equation of the fourth or eighth order: in each case the equation being of a lower order when the difference of phase in the two movements is zero or 180°.

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* Memoirs Amer. Acad., 1st series, vol. iii. p. 413.

Dr. Bowditch adds: "I made a few experiments in order to compare the preceding theory with observation." In his first experiment, the two points of suspension were 4.75 inches apart, the single branch of the thread was 46.5 inches, and the *vertical* height of the double branch only .65 of an inch. Therefore the two movements were more nearly in unison than in Professor Dean's experiment. A ball of lead about half an inch in diameter hung at the bottom. By calculation one movement would gain a whole vibration upon the other in 286 vibrations of the more rapid movement: by experiment the number came out 282. In another experiment, the two points of suspension were 69 inches asunder: the single branch of the pendulum was 21.9 inches, and the vertical height of the double branch 64.4 inches. A ball of lead $1\frac{2}{3}$ inches in diameter was attached. In this case the two movements differed in period by the interval of an octave, imperfect to the extent of one vibration in sixty-seven of the slower movement; and experiment confirmed this conclusion, the whole cycle of orbits being completed in that time. In regard to the two other ratios, Dr. Bowditch says: "A few rough experiments were, however, made in these cases, and the results appeared to be sufficiently conformable to the theory." This kind of experiment is not suited to such ratios as 1 : 3 and 1 : 4. For in these cases the single branch of the cord is only one ninth and one sixteenth of the total length, and the amplitude of its vibration comparatively small.

I have now traced the mathematical analysis and the experimental illustration of the Lissajous curves from France to Great Britain, and thence, across the ocean, to their home in Salem, Massachusetts. The so-called Lissajous curves are the Bowditch curves, except so far as the earth itself had been experimenting upon one set of them for thousands of years. They will continue, probably, to be called the Lissajous curves. But their history should be known, and will be known; though it is not necessary for the reputation of the self-taught mathematician, Dr. Nathaniel Bowditch. The author of the "Practical Navigator," and the translator of Laplace's "Mécanique Céleste," with its rich commentary, has secured a place in the world of commerce and in the world of science to which nothing need be or can be added.

Explanation of the Plate.

Figure 1. Professor Dean's diagram, illustrating his pendulum.

Figures 2, 3, 4, and 5. Dr. Bowditch's diagrams to show the orbits in an imperfect unison.

Figure 6. Dr. Bowditch's diagram to illustrate the progressive changes in the orbit.

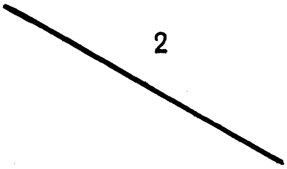
Figures 7, 8, 9, 10 and 11. Dr. Bowditch's diagram to show the orbits in an imperfect octave.

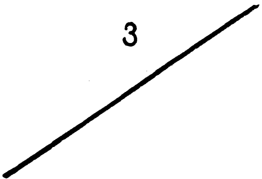
Figures 12 and 13. Dr. Bowditch's diagram for special differences of phase in the ratio, 1 : 3.

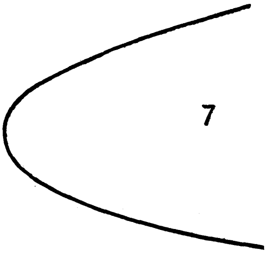
Figures 14 and 15. The same when the ratio was 1 : 4.

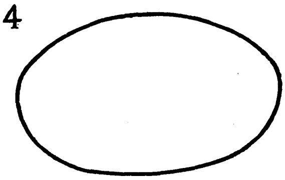
Figure 16. General diagram of Dr. Bowditch's pendulum.

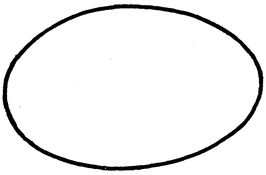
All these figures are exact copies of those published in the Plates of the Academy, vol. iii. 1st series, 1815.

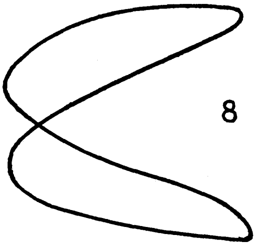


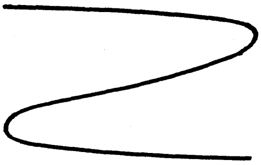


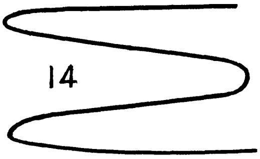


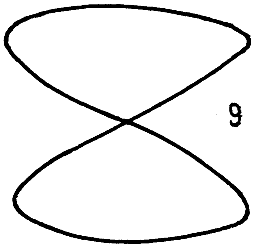


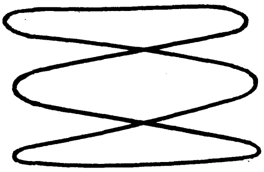


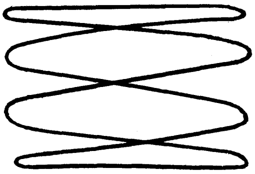


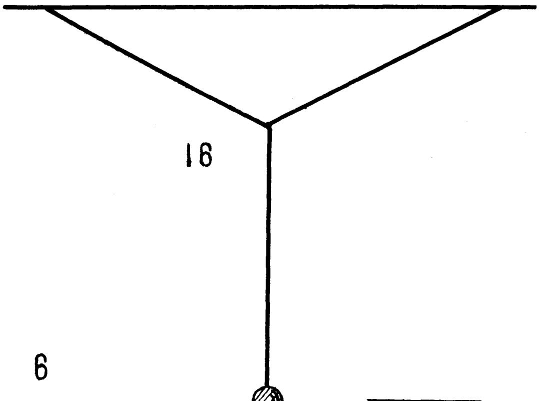


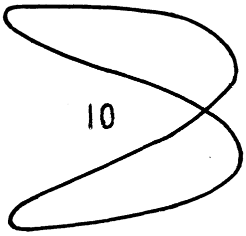


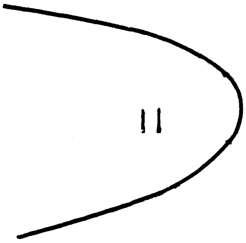


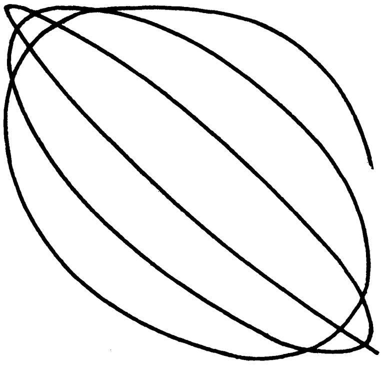


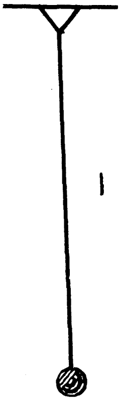












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